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APPLICATION OF THE AVERAGING METHOD IN THE CALCULATION OF ROD STRUCTURES LIKE PLATES AND BEAMS

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A modification of the averaging method that permits one to calculate the averaged characteristics of periodic beam structures such as masts, skeleton spans., etc. by the methods of materials strength is proposed.

Introduction. As the inverse problem, the problems of description of rod structures such as masts, skeleton spans, etc. with the use of continuous models [1] have been raised in the literature owing to the development of theoretical and computational methods. The state of the problem up to 1988 was reported by Noor [2] and Pshenichnov [3]. Recently, the problem has been attacked by homogenization (averaging) method, which appeared in the 1970s [4–7].

The results obtained in the late 1980s [8–11] allow one to develop an engineering theory of rod-structure calculation by means of homogenization [4, 5] and classical methods of materials strength. Annin et al. [12] used this combination of methods to calculate plate-containing structures.

The basic element of the homogenization method is the solution of the so-called cell problem. In [8, 9], the author reduced this problem to the plate/beam problem, and the arising errors were estimated. The interest in the problems of averaged description of rod structures makes it expedient to further develop the computational methods based on the averaging theory. This study is devoted to a construction of models of averaged description of small-thickness structures like inhomogeneous plates and beams [13, 14]. Figure 1 gives examples of structures similar to a plate (a) and a beam (b); periodicity cells (PC) in these structures are plotted on the right (P_1 are the regions occupied by the material, m is the period, and γ is the lateral surface of the PC).

1. Formulation of the Problem. The major result of the mathematical averaging theory [4, 5] is the proof that an inhomogeneous periodic medium, in particular, a void-containing medium like the structures considered, can be put into correspondence with a homogeneous body close in its mechanical behavior if the characteristic size of the PC of the inhomogeneous body is $\varepsilon \ll 1$ (Fig. 1). Caillerie [13] generalized these results to plates, and the author [14] to beams. We note that the parameter ε also characterizes the thinness of the structure walls.

The correspondence between the local elastic constants of an inhomogeneous medium $a_{ijkl}(\mathrm{y})$ and its averaged constants is established as follows [12–14]. We solve the following CP

$$\begin{gathered}\frac{\partial}{\partial y_j}\Big(a_{ijkl}(\mathrm{y})\,\frac{\partial N_k^{\nu\alpha}}{\partial y_l}+a_{ij11}(\mathrm{y})y_\alpha^\nu\Big)=0 \text{ in } P_1,\\ \Big(a_{ijkl}(\mathrm{y})\,\frac{\partial N_k^{\nu\alpha}}{\partial y_l}+a_{ij11}(\mathrm{y})y_\alpha^\nu\Big)n_j=0 \text{ on } \gamma\end{gathered} \tag{1.1}$$

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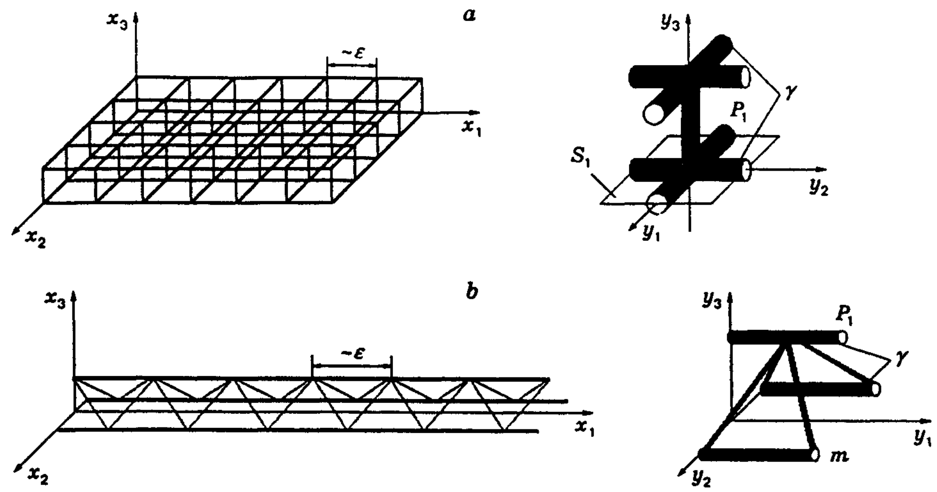

Fig. 1

for beams, where $N^{\nu\alpha}(y)$ is periodic in y_1 with period m and $\langle N^{\nu\alpha}\rangle = 0$, and

$$\frac{\partial}{\partial y_j}\left(a_{ijkl}(y)\frac{\partial N_k^{\nu\alpha\beta}}{\partial y_l} + a_{ij\alpha\beta}(y)y_3^{\nu}\right) = 0 \text{ in } P_1,$$
$$\left(a_{ijkl}(y)\frac{\partial N_k^{\nu\alpha\beta}}{\partial y_l} + a_{ij\alpha\beta}(y)y_3^{\nu}\right) = 0 \text{ on } \gamma \qquad (1.2)$$

for plates, where $N^{\nu\alpha\beta}(y)$ is periodic in y_1 and y_2 and has the PC S_1 and $\langle N^{\nu\alpha\beta}\rangle = 0$ and $\nu = 0,\ 1$. Here P_1 is the PC in the dimensionless coordinates $y = x/\varepsilon$, γ is its lateral surface on which the periodicity condition is not imposed (Fig. 1), $L_1 = [0, m]$ is the projection of the PC onto the Oy_1 axis for the beam, S_1 is the projection of the PC onto the Oy_1y_2 plane for the plate, $\langle\cdot\rangle = \frac{1}{\text{mes}S_{\text{pr}}}\int_{P_1}\cdot\, dy$ is the average over the PC P_1, and $S_{\text{pr}} = L_1$ or S_1.

After the cell problem is solved, the characteristics of the homogeneous body are calculated by the following formulas:

$$S_{\alpha\beta}^{\nu+\mu} = \left\langle\left(a_{1111}(y)y_\alpha^{\nu} + a_{11kl}(y)\frac{\partial N_k^{\nu\alpha}}{\partial y_l}\right)y_\beta^{\mu}\right\rangle \qquad (1.3)$$

for beams [9] and

$$S_{\alpha\beta\gamma\delta}^{\nu+\mu} = \left\langle\left(a_{\alpha\beta\gamma\delta}(y)y_3^{\nu} + a_{\alpha\beta kl}(y)\frac{\partial N_k^{\nu\gamma\delta}}{\partial y_l}\right)y_3^{\mu}\right\rangle \qquad (1.4)$$

for plates [8].

The superscripts ν and μ take the values 0 and 1; the subscripts α, β, γ, and δ take the values 1 and 2 for beams and 2 and 3 for plates. Here $S^0_{\dots}$ are the tensile–compressive rigidities, $S^1_{\dots}$ are the lateral rigidities, and $S^2_{\dots}$ are the flexural–torsional rigidities of the averaged structures (beams and plates).

Remark. For a homogeneous cylindrical rod, the cell problem (1.1) and formula (1.3) are reduced to those obtained in [7]. For cellular shells, this approach yields the results obtained in [3] (see [12]).

2. Reduction of the Cell Problem to Problems of Materials Strength. Problems (1.1) and (1.2) are the particular problems of the theory of elasticity. The specifics of these problems consists in the presence of constant terms of the form $(\partial/\partial y_j)(a_{ij\alpha\beta}(y)y_\delta^{\nu})$ and periodic boundary conditions. These conditions cannot be transformed to the standard ones, except the particular cases of a symmetric PC. However, this does not raise problems, because the periodicity conditions can be reformulated in terms of the theory of materials strength.

In solving the cell problem, the difficulties are arisen by constant terms which correspond to nontypical mass forces; the latter can be eliminated for $\nu = 0$ by a known replacement [4, 5]

$$\mathbf{W} = \mathbf{N}^0 + y_1 \mathbf{e}_1$$

for beams and (2.1)

$$\mathbf{W} = \mathbf{N}^{0\alpha\beta} + y_\alpha \mathbf{e}_\beta$$

for plates. As a result, for displacements $\mathbf{W}$ we obtain, from (1.1) and (1.2), the standard equations of the theory of elasticity without mass forces.

We show that a replacement similar to (2.1) can be made for $\nu = 1$ (for the flexure–torsion problem) as well. For this purpose, it suffices to check that there are the displacements $\boldsymbol{\xi}$ such that the following equalities are fulfilled for the strains $e_{kl} = (1/2)(\partial\xi_k/\partial y_l + \partial\xi_l/\partial y_k)$:

$$a_{ijkl}(\mathbf{y})e_{kl} + a_{ij11}(\mathbf{y})y_\beta = 0$$

for beams and (2.2)

$$a_{ijkl}(\mathbf{y})e_{kl} + a_{ij\alpha\beta}(\mathbf{y})y_3 = 0$$

for plates, or, which is equivalent to (2.2),

$$e_{kl} = -c_{klij}(\mathbf{y})a_{ij11}(\mathbf{y})y_\beta$$

for beams and (2.3)

$$e_{kl} = -c_{klij}(\mathbf{y})a_{ij\alpha\beta}(\mathbf{y})y_3$$

for plates, where $\{c_{ijkl}\}$ is the rigidity tensor inverse to $\{a_{ijkl}\}$.

It follows from (2.3) that

$$e_{kl} = -\delta_{k1}\delta_{l1}y_\beta$$

for beams and (2.4)

$$e_{kl} = -\delta_{kl}\delta_{l\beta}y_3$$

for plates. Here $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

It is possible to verify that the strains (2.4) satisfy the classical compatibility conditions [15].

Thus, the unknown displacements $\boldsymbol{\xi}$ exist. Below, we write explicitly for particular cases.

We denote by $\boldsymbol{\xi}^{\nu\alpha\beta}$ the following functions

$$\boldsymbol{\xi}^{\nu\alpha\beta} = \boldsymbol{\xi}^{\nu\alpha}, \quad \boldsymbol{\xi}^{0\alpha} = y_1\mathbf{e}_1, \quad \boldsymbol{\xi}^{1\beta} = \boldsymbol{\xi} \tag{2.5}$$

for beams and

$$\boldsymbol{\xi}^{0\alpha\beta} = y_\alpha\mathbf{e}_\beta, \quad \boldsymbol{\xi}^{1\alpha\beta} = \boldsymbol{\xi} \tag{2.6}$$

for plates. It is noteworthy that the function $\boldsymbol{\xi}$ depends on β for beams and on α and β for plates.

Using the functions (2.5) and (2.6), one can write uniformly the cell problem (1.1) and (1.2) as

$$\frac{\partial}{\partial y_j}\left(a_{ijkl}(\mathbf{y})\,\frac{\partial W_k}{\partial y_l}\right) = 0 \text{ in } P_1, \quad a_{ijkl}(\mathbf{y})\,\frac{\partial W_k}{\partial y_l}\,n_j = 0 \text{ on } \gamma, \tag{2.7}$$

where the function $\mathbf{W} - \boldsymbol{\xi}$ is periodic in $\mathbf{y} \in S_{\rm pr}$, $\langle \mathbf{W} - \boldsymbol{\xi} \rangle = 0$, and

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\nu\alpha\beta}. \tag{2.8}$$

In solving the problems, it is necessary to bear in mind the definitions (2.5) and (2.6).

3. Method of Solving the Cell Problem (2.7). Problem (2.7) is the typical problem of the theory of elasticity which is reducible to the beam/rod problems [8–11]. The author showed [8, 9] that to it corresponds the beam problem written under the following conditions:

(1) the mass and surface forces are equal to zero;

(2) the rigid-joint conditions are satisfied in the inner nodal joints of the beams and the sums of the forces and moments in the nodal joint are equal to zero;

(3) in the boundary nodal joints, the periodicity conditions $\mathbf{W}-\boldsymbol{\xi}$ are satisfied for the displacement of the corresponding forces and moments.

We mean the nodal joints of beams/rods by inner nodes and the nodal joints lying on the PC surfaces on which the periodicity conditions are imposed by the boundary nodes.

The results of [8–11] were obtained for a three-dimensional (not thin) structure. However, since the cell problem (2.7) is unique for all cases [for a three-dimensional composite, a thin structure, and a small-diameter, only the function $\boldsymbol{\xi}^{\nu\alpha\beta}$ in (2.8) varies], all the results are extended to the last two cases according to the same considerations as in [8–11]. Some difficulties arise in the presence of hinged joints to which the problem of elasticity in the form described above does not correspond. Hinged joints can be described within the framework of contact problems with the use of variational inequalities. For such problems, the question of averaging was solved positively [16, 17]. Consideration of the cell problem for a cellular structure with hinged joints leads to a problem with conditions (1) and (3), where the following condition is used instead of condition (2): the connection conditions are satisfied in the inner nodes for displacements and forces in the absence of moments.

The behavior of beams and rods can be described in terms of generalized displacements [18] of their ends ($\mathbf{u}_+$ and $\mathbf{u}_-$).

The axial and cutting forces $\mathbf{N}$ and $\mathbf{Q}$ and the moments M_α relative to the Oy_α axis in a beam/rod in the absence of mass forces [see condition (1)] are determined by generalized displacements. This dependence can be written both in the local (connected to a beam/rod) and in the $y_1y_2y_3$ coordinate system.

The connection conditions in the inner points [see condition (2)] can be satisfied by introducing the generalized displacements of the nodes.

The periodicity condition $\mathbf{W}-\boldsymbol{\xi}$ [see condition (3)] also can be formulated in terms of the generalized displacements of the nodes:

$$\mathbf{u}_i = \mathbf{u}_{\Gamma(i)}, \tag{3.1}$$

where i is a node that belongs to the PC edge; $\Gamma(i)$ is the corresponding node belonging to the opposite edge, and $\{\mathbf{u}_i\}$ are the generalized displacements which correspond to $\mathbf{W}-\boldsymbol{\xi}$.

For inner nodes, the conditions for forces and moments [see condition (2)] are of the form

$$\sum_{j\in K_i}[\mathbf{N}(\mathbf{u}_i,\mathbf{u}_j)+\mathbf{Q}(\mathbf{u}_i,\mathbf{u}_j)]=0, \quad \sum_{j\in K_i} M_\alpha(\mathbf{u}_i,\mathbf{u}_j)=0, \tag{3.2}$$

where K_i are the nodes jointed to the node i by the structural elements.

The periodicity conditions for forces and moments [see condition (3)] in the boundary points have the form (3.2) if, for the node i lying at the PC edge, the K_i is replaced by

$$K_i \cup K_{\Gamma(i)}. \tag{3.3}$$

The displacements $\mathbf{W}$ are determined from (3.1)–(3.3) to within a displacement of a solid. Therefore, the condition $\langle \mathbf{W}-\boldsymbol{\xi}\rangle = 0$ can be replaced by $\langle \mathbf{W}\rangle = 0$ and is written as follows:

$$\sum_{i=1}^{m}\mathbf{u}_i = 0 \tag{3.4}$$

(m is the total number of nodes). Condition (3.4) does not arise from the calculation of the integrals in the equality $\langle \mathbf{W}\rangle = 0$ but is equivalent to this equality.

Equations (3.1)–(3.4) represent an algebraic system of the form

$$T\{\mathbf{u}_i\} = \mathbf{b}. \tag{3.5}$$

System (3.5) is the cell problem written in terms of the theory of materials strength.

4. Calculation of Averaged Rigidities. Formulas (1.3) and (1.4) assume integration over the structural elements of the PC. Integration can be performed by representing the solutions of the cell problem via the generalized displacements $\{\mathbf{u}_i\}$. However, it is possible to derive expressions (1.3) and (1.4) via stresses at the PC edges, which are similar to the expressions for homogeneous plates and beams. We shall show how it can be done. We denote

$$\begin{aligned} \sigma_{ij}^{\nu\alpha} &= a_{ij11}(\mathbf{y})y_\alpha^\nu + a_{ijkl}(\mathbf{y})\frac{\partial N_k^{\nu\alpha}}{\partial y_l} \quad \text{for beams}, \\ \sigma_{ij}^{\nu\alpha\beta} &= a_{ij\alpha\beta}(\mathbf{y})y_3^\nu + a_{ijkl}(\mathbf{y})\frac{\partial N_k^{\nu\alpha\beta}}{\partial y_l} \quad \text{for plates.} \end{aligned} \tag{4.1}$$

We note that the quantities $\sigma_{ij}^{\nu\alpha}$ and $\sigma_{ij}^{\nu\alpha\beta}$ are equal to $a_{ijkl}(\partial W_k/\partial y_l)$, where $\mathbf{W}$ is the solution of problem (2.7).

The PC is considered (which does not limit the generality) equal to $[-1/2, 1/2]$ for beams and to $[-1/2, 1/2] \times [-1/2, 1/2]$ for plates.

For the stresses σ_{ij} (by σ_{ij} we mean $\sigma_{ij}^{\nu\alpha}$ or $\sigma_{ij}^{\nu\alpha\beta}$ depending on the problem considered), the cell problem can be written as follows:

$$\frac{\partial \sigma_{ij}}{\partial y_j} = 0 \quad \text{in} \quad P_1, \qquad \sigma_{ij}n_j = 0 \quad \text{on} \quad \gamma. \tag{4.2}$$

We shall present the method of calculating the quantities $S_{\alpha\beta}^{1+\nu}$ for a beam. We multiply the equations in (4.2) by $y_\beta y_1$ ($\beta = 2, 3$) and integrate by parts the result. With allowance for the boundary condition in (4.2), we obtain

$$-\langle \sigma_{i1}^{\nu\alpha} y_\beta \rangle - \frac{1}{\text{mes}S_{\text{pr}}}\int\limits_{P_1} \sigma_{i\beta}^{\nu\alpha} y_1 \, d\mathbf{y} + \frac{1}{\text{mes}S_{\text{pr}}}\int\limits_{\Gamma_+\cup\Gamma_-} \sigma_{i1}^{\nu\alpha} y_1 y_\beta \, d\mathbf{y} = 0. \tag{4.3}$$

Here $\langle \sigma_{11}^{\nu\alpha} y_\beta \rangle$ is equal to $S_{\alpha\beta}^{1+\nu}$ [see (1.3) and (4.1)] and Γ_+ and Γ_- are the opposite edges of the PC (Fig. 1). Thus $\sigma_{i1}^{\nu\alpha}$ are periodic, $y_1 = \pm 1/2$ on $\Gamma_\pm$, and $\text{mes}S_{\text{pr}} = 1$. Then the integral in the last term in (4.3) is equal to

$$\int\limits_{\Gamma_+} \sigma_{i1}^{\nu\alpha} y_\beta \, d\mathbf{y} = \sum_{i\in\Gamma_+} [(\mathbf{N}^{\nu\alpha} + \mathbf{Q}^{\nu\alpha})_{1i} y_{\beta i} + M_{\beta i}^{\nu\alpha}], \tag{4.4}$$

where $(\mathbf{N}^{\nu\alpha} + \mathbf{Q}^{\nu\alpha})_{1i}$ is the projection of forces in the ith boundary node onto the Oy_1 axis, $y_{\beta i}$ is the coordinate of the ith boundary node along the Oy_β axis, and $M_{\beta i}^{\nu\alpha}$ is the moment in the ith boundary node ($\beta = 2, 3$).

In the ith boundary node, we have

$$\mathbf{N}^{\nu\alpha} = \sum_{j\in K_i} \mathbf{N}(\mathbf{u}_i, \mathbf{u}_j), \quad \mathbf{Q}^{\nu\alpha} = \sum_{j\in K_i} \mathbf{Q}(\mathbf{u}_i, \mathbf{u}_j), \quad M_{\beta i}^{\nu\alpha} = \sum_{j\in K_i} M_\beta(\mathbf{u}_i, \mathbf{u}_j). \tag{4.5}$$

The integral over P_1 in (4.3) is zero. To be convinced ourselves, we multiply the equations in (4.2) by y_1^2 and integrate by parts the result. With allowance for the boundary condition, from (4.2) we obtain

$$-2\int\limits_{P_1} \sigma_{i1}^{\nu\alpha} y_1 \, d\mathbf{y} + \int\limits_{\Gamma_+\cup\Gamma_-} \sigma_{i1}^{\nu\alpha} y_1^2 \, d\mathbf{y} = 0. \tag{4.6}$$

In the last integral in (4.6), both functions σ_{i1} and y_1^2 are periodic in y_1. By virtue of this periodicity, the integral over $\Gamma_+ \cup \Gamma_-$ equals zero, and thus we arrive at our statement for $i = \beta$.

Then we obtain from (4.3) and (4.4) the equality

$$S^{1+\nu}_{\alpha\beta} = \sum_{i\in\Gamma_+} [(\mathbf{N}^{\nu\alpha} + \mathbf{Q}^{\nu\alpha})_{1i} y_{\beta i} + M^{\nu\alpha}_{\beta i}]. \tag{4.7}$$

Similarly, it is possible to derive the expression

$$S^{0+\nu}_{\alpha\beta} = \sum_{i\in\Gamma_+} (\mathbf{N}^{0\alpha} + \mathbf{Q}^{0\alpha})_{1\alpha}$$

with the use of the equality [13]

$$\int_{P_1} \sigma^{\nu\gamma\delta}_{\beta 1} y_1 \, dy = 0.$$

For plates, the expression $S^{\mu+\nu}_{\alpha\beta\gamma\delta}$ via the boundary values of forces and moments can be derived by multiplying Eq. (4.2) by $y_3 y_\alpha$ and by integrating by parts the results. The calculations are similar to those mentioned above. The formulas for $S^{1+\nu}_{\alpha\beta\gamma\delta}$ and $S^{0+\nu}_{\alpha\beta\gamma\delta}$ coincide with (4.7), but it is necessary to take into account, that σ_{ij} and **N**, **Q**, and M_α have the corresponding subscripts. We note that **N**, **Q**, and M_α are determined via $\{\mathbf{u}_i\}$, which are the solutions of problem (3.5).

5. Rod Structures. We consider a rod cellular structure. The elements of the cellular structure work only in tension–compression. Then we have $Q = 0$, $M_\beta = 0$, and the axial forces in a rod are as follows:

$$\mathbf{N} = E(\mathbf{u}_+ - \mathbf{u}_-, \mathbf{l})\mathbf{l}/|\mathbf{l}|, \tag{5.1}$$

where E is the rigidity of the rod, $\mathbf{l}$ is its directing vector, and $|\mathbf{l}|$ is the length of the vector.

With allowance for (5.1), Eqs. (3.1) take the form

$$\sum_{j\in K_i} E_{ij}(\mathbf{u}_i - \mathbf{u}_j, \mathbf{l}_{ij})\mathbf{l}_{ij}/|\mathbf{l}_{ij}| = 0, \tag{5.2}$$

where the subscript ij refers to the rod from the ith to the jth node and E_{ij} is the rigidity of this rod.

For the case considered, from (4.7) we obtain

$$S^{1+\nu}_{\alpha\beta} = \sum_{i\in\Gamma_+} N^{\nu\alpha}_{1i} y_{\beta i}, \qquad S^{0+\nu}_{\alpha\beta} = \sum_{i\in\Gamma_+} N^0_{1i}. \tag{5.3}$$

The solution (2.4) has the form [19]

$$\xi_1 = y_1 y_2, \quad \xi_2 = (1/2)y_1^2 + y_3, \quad \xi_3 = -y_2$$

for a beam considered as a three-dimensional body and

$$\xi_1 = y_1 y_3, \quad \xi_3 = -(1/2)y_3^2 \tag{5.4}$$

for a beam considered as a plane body.

6. Examples of the Calculation of Averaged Rigidities. *Homogeneous Beam.* Let the PC consist of one element lying on the Oy_1 axis. We have $y_3 = 0$, $Q = 0$, $N_{11} = E$, and $M_1^{\nu 3} = D$, where E and D is the tensile and flexural rigidities of the beam.

A Beam with an I-shaped PC. A plane I-shaped PC is shown in Fig. 2 (the deformed PC is shown by the dashed curve). By virtue of symmetry of the PC and the presence of symmetry in the function (5.4), the strain of the PC is reduced to tension (compression) of the upper (lower) beam of the PC and to displacement of the PC along the axis as a solid. With allowance for formula (5.4), we obtain $Q^1 = 0$, $M^{13} = 0$, and

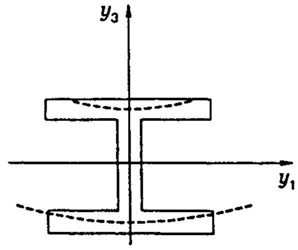

Fig. 2

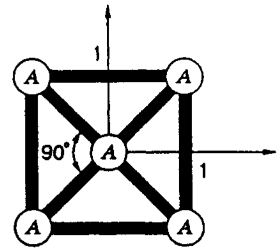

Fig. 3

$N_i^{13} = \pm(1/2)E$ for the upper (lower) beam, and $y_{3i} = \pm(1/2)H$. According to (4.7), the flexural rigidity equals

$$S_{33}^2 = \sum_{i=1}^{2} N_{1i}^{13} y_{3i} = 2E\Big(\pm\frac{1}{2}\Big)\Big(\pm\frac{1}{2}H\Big) = \frac{1}{2}HE.$$

We consider the planar cellular structure depicted in Fig. 3 ($P_1 = [-1,1] \times [-1,1]$) as a rod structure when A are the hinges and as a beam structure when A are the rigid joints.

Rod Structure. With allowance for symmetry, the problem is reduced to a calculation of strains with the PC nodes displaced in the direction of the Oy_1 axis. The upper nodes are displaced from each other, and the lower nodes to each other. The forces in the upper and lower horizontal rods equal $\mp(1/2)E$. With allowance for $y_3 = \pm 1$, from (5.3) we obtain the flexural rigidity: $S_{33}^2 = E_1$.

Beam Structure. It is necessary to solve the tension–flexure problem of a system of beams. From the solution of this problem and from (4.7), it follows the expression for flexural rigidity

$$S_{33}^2 = E_1 + 6(\sqrt{2}+1)\,\frac{D}{6D/E + 1/\sqrt{2}}. \tag{6.1}$$

Here E_1 is the tensile rigidity of the horizontal beams, which is assumed to be equal for both beams, and E and D are the tensile and flexural rigidities of the sloping beams, which also are assumed to be equal. As is seen, if the flexural rigidity is small ($D \ll E$), one can ignore the last term in (6.1) in comparison with E_1 for E and E_1 of the same order.

7. Thermoelastic Characteristics of the Beams. The problem of thermoelasticity was considered for three-dimensional composite bodies in [20, 21], for plates in [22–24], and for beams in [25]. The cell problem of thermoelasticity has the form [22–25]

$$\frac{\partial}{\partial y_j}\Big(a_{ijkl}(\mathbf{y})\,\frac{\partial N_k^0}{\partial y_l} + \beta_{ij}(\mathbf{y})\Big) = 0 \quad \text{in} \quad P_1,$$

$$\Big(a_{ijkl}(\mathbf{y})\,\frac{\partial N_k^0}{\partial y_l} + \beta_{ij}(\mathbf{y})\Big)n_j = 0 \quad \text{on} \quad \gamma,$$

where $\mathbf{N}^0(\mathbf{y})$ is a periodic function [see (1.1) and (1.2)].

The thermoelastic characteristics are calculated by the formulas

$$\alpha_\beta^\nu = \Big\langle\Big(\beta_{11}(\mathbf{y}) + a_{11kl}(\mathbf{y})\,\frac{\partial N_k^0}{\partial y_l}\Big)y_\beta^\nu\Big\rangle$$

for beams and

$$\alpha_{\alpha\beta}^\nu = \Big\langle\Big(\beta_{\alpha\beta}(\mathbf{y}) + a_{\alpha\beta kl}(\mathbf{y})\,\frac{\partial N_k^0}{\partial y_l}\Big)y_3^\nu\Big\rangle$$

for plates. The superscript ν takes the values 0 and 1 (for the values of the subscripts α and β see in Sec. 1).

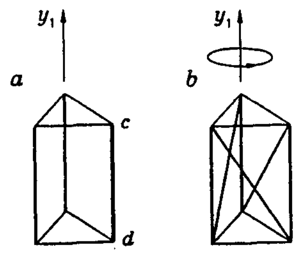

Fig. 4

Introducing the stresses $\sigma_{ij} = \beta_{ij}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\partial N_k^0/\partial y_l$, we obtain the cell problem in the form (4.2). Here, the $\mathbf{N}^0$ can be interpreted as displacements.

As a result, we obtain that α_β^0, α_β^1, $\alpha_{\alpha\beta}^0$, and $\alpha_{\alpha\beta}^1$ are specified by formulas (4.7), in which **N**, **Q**, and M_α are determined from the solution of the cell problem of thermoelasticity. This problem is formed on the basis of conditions (1)–(3). The quantities **N**, **Q**, and M_α for the cellular-structure members are found from the governing relations of thermoelastic rods/beams:

$$N = E(\varepsilon - \beta\theta), \quad M = D(\rho - \beta^*\theta),$$

where E and D are the tensile and flexural rigidities and β and β^* are the coefficients of thermal (axial and flexural) expansion. For homogeneous cylindrical beams, we have $\beta^* = 0$.

8. Thermoelastic Rod Beams and Control of Their Characteristics. We consider a small-diameter structure formed from rods. The governing equations for this structure as being a one-dimensional beam, are as follows [13]:

$$N = A^0\varepsilon + A_\alpha^1\rho_\alpha + B_0\varphi + C\theta, \qquad M_\alpha = A_\alpha^1\varepsilon + A_{\alpha\beta}^2\rho_\beta + B_\alpha\varphi + C_\alpha\theta,$$
$$\mathcal{M} = A_\varepsilon + A_\alpha^m\rho_\alpha + B\varphi + C^m\theta, \tag{8.1}$$

where N, M_α, and $\mathcal{M}$ are the axial force, the flexural moments, and the torsional moment, respectively.

As follows from (8.1), the temperature can cause all kinds of deformation : axial elongation–compression, flexure, and torsion.

Equations (8.1) can be derived from (5.3). The temperature enters (5.3) implicitly when (5.2) is replaced by the equations

$$\sum_{j\in K_j} E_{ij}((\mathbf{u}_i - \mathbf{u}_j, \mathbf{l}_{ij})/|\mathbf{l}_{ij}| - \beta)\mathbf{l}_{ij} = 0.$$

With a variable beam structure (the connections $\{K_i\}$ and the rigidities $\{E_{ij}\}$), it is possible to formulate the problem of control: to assign specified values to the averaged rigidities in (8.1) owing to the choice of the characteristics of the structural elements of the beam. We shall give several examples of qualitative character, which show the possible results of control.

A PC can consist of several types of structures (we call them circles) located on the y_1 axis (Fig. 4). Below, we illustrate circles with distinct mechanical properties:

(1) The section is uniformly elongated during heating in the direction of the structure's axes (all the vertical rods are identical) (Fig. 4a);

(2) The section is extended during heating and is curved in the direction of the element c–d, which has a different coefficient of thermal expansion (Fig. 4a) compared to other elements;

(3) The section is extended during heating and is twisted about the y_1 axis (Fig. 4b). The circles' members made from different materials are shown in Fig. 4 by the lines of different thickness.

Circles of the cited types are connected with each other. The three-circle cellular structure obtained imparts the averaged beam the ability to extend, bend, and twist during heating.

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